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1982 J. Phys. A: Math. Gen. 15 1991

(http://iopscience.iop.org/0305-4470/15/7/011)

J. Phys. A: Math. Gen. **15** (1982) 1991–2002. Printed in Great Britain

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Received 18 March 1981, in final form 15 March 1982

Abstract. A listing is proposed for the steps in the logic of dimensional analysis. Starting with the definition of a physical concept, the arguments proceed through the principles of measurement, by the specification of dimensional equality and the subsequent limitations of functional operations, up to a derivation of the Pi-theorem of this analysis.

1. Introduction

In 1931, Bridgman, in the second edition of his book on dimensional analysis, drew attention to the continued existence of 'differences in fundamental points of view'. He repeated this view in 1959 by pointing out that there had 'not by any means been agreement with regard to the philosophy of the subject, and many questions are still controversial'. His final contribution would appear to have been the joint one with Sedov in 1974; in it uncertainties, to be considered here, were still expressed.

A previous paper (Gibbings 1980) contained a discussion of only some of the steps in the logic of dimensional analysis, without ordering those steps, and had accepted and used the results of Buckingham's Pi-theorem (1914). The present paper lists and discusses, in a necessarily condensed form, a proposed complete order of logic up to and including the Pi-theorem.

In summary, the order of logic to be advanced here is now listed in its successive stages.

(i) The concept of a 'primary' physical quantity is defined.

(ii) The unit reference quantity is defined.

(iii) The measure of a 'primary' quantity is defined as addition of unit quantities.

(iv) The measure by addition results in a linear scale.

(v) The existence of linear scales together with common origins results in the constancy of relative magnitude.

(vi) The concept of a 'derived' physical quantity is defined through either an arbitrary defining relation or an observed physical law.

(vii) The definition of a 'derived' quantity being in terms of products of quantities—to include division and the raising to powers—then dimensions are combined also in the same products.

(viii) The constancy of relative magnitude is then retained in such products; definition of 'derived' quantities not in terms of products is not accepted.

(ix) The dimensions in products of quantities can be cancelled.

(x) Dimensional equality is specified for numerical addition of quantities.

(xi) The arbitrary choice of unit quantity for a 'derived' quantity requires the introduction of a 'units conversion factor' to retain dimensional equality.

(xii) The concept of the 'complete' equation follows from the specification of dimensional equality.

(xiii) The retention of a 'complete' equation limits the functional operations that can be performed on it.

(xiv) The appearance of 'unit conversion factors' in an equation is directly related to the inclusion of the corresponding defining relations in the analytical model.

(xv) The foregoing of an arbitrary unit quantity for a derived quantity in an equation results in the removal of the corresponding 'units conversion factor'.

(xvi) Variables in an equation can be grouped in products, the limit to this grouping occurring when the products become non-dimensional.

By no means are all these steps in the logic new. The purpose of this paper is to present this particular ordering and totality of the logic and to clarify some of the steps.

2. Quantifying observation

In science, concern is with observation of material things forming a system, as that term is defined in a generalised thermodynamic sense (Gibbings 1970), and in the way that they behave during a process. Quantification of the state of a system is by quantification of its properties and these properties derive from concepts. Then a first step is to define concepts.

Two types of concept are proposed. Following a prior discussion (Gibbings 1980), concepts such as length and time, which in that discussion are defined in terms of what can be comprehended by the senses in a fundamental way, might be thought of as 'primary' concepts. In the same manner the concept of force could be defined as being sensed by a difference of loading, such as by holding a system against gravitational attraction. A unit quantity for each of these three 'primary' concepts can be defined quite independently of each other. For length it might be a dimension of a certain piece of material; for time it might be the interval of one swing of a certain pendulum; for force it might be the load to fully compress a certain spring. The measure of each of these 'primary' concepts is then specified as being by addition of these unit quantities: following the previous discussion (Gibbings 1980), the definition of measure of these 'primary' concepts is separate from, though consequent upon, the definition of the concept. In this approach we follow Bacon (1868) who wrote 'And it is a grand error to assert that sense is the measure of things'.

Because of the independence of the 'unit quantities' it follows in principle that these three 'primary' concepts, length, time and force, can be measured without any reference to each other. This independence of measurement was advanced by Esnault-Pelterie (1950) as the definition of a 'primary' concept, rather than as here where the initial definition is as a sensed one. Here again we follow Bacon (1840) who wrote 'For information begins with the senses. But our whole work ends in Practice . . . '. Also definition by the idea of independence of measurement does not seem to accord with the choice by Esnault-Pelterie of mass, rather than force as chosen here, as the third primary concept.

The measure of 'primary' concepts by addition of 'unit quantities' results in a linear scale of measurement with a scale zero at zero amount. It then follows, as Bridgman (1943) pointed out, that the ratio of two quantities of the same concept is independent of the size chosen for the 'unit quantity'; there is a constancy of relative magnitude.

From the definition of the measure of a primary concept, a multiple of the smallest distinguishable quantity is chosen as a convenient unit quantity. In principle, all other quantities are then taken as the arithmetical count of the content of smallest quantities.

So far the first five logical steps listed have been considered.

In contrast to 'primary' concepts, a concept such as velocity does not seem to be comprehensible in such a fundamental way, but is what might be called a 'derived' concept as its definition is in terms of the ratio of measures of the 'primary' concepts of length and of time. Similarly other 'derived' concepts can be defined in succession as shown in table 1†, all as definitions of the measure of them in terms of the measure of other concepts‡.

Table 1.

		Number in defining relation			
		Measures			
No	Defining relation(s)	Primary	Derived	Constants	SI unit
1	$V = s/t$	2			
2	$\alpha = (1/\beta_0)s/r$	1		1	α
3	$E_r = h_p f$	1		1	
4	$Q = (1/J)Fs$	2		1	
5	$F = \gamma\Gamma^2/r^2$	2		1	
6	$F = q^2/\varepsilon_0 r^2$	2		1	
7	$a = V/t$	1	1		
8	$i = q/t$	1	1		A
9	$F = g_0 ma$	1	1	1	M
10	$E_k = \frac{1}{2}mV^2$		2		
11	$\frac{1}{2}m\bar{c}^2 = k_B T$		2	1	θ
12	$\lambda = E_l/P_0 : E_l = E_k$		2	1	C
13	$\mu_0 H = F/si$	2	1	1	
14	$p/\rho = (R/M_0)T : Q = m/M_0$	2	2	1	n

The defining relations are listed in this table and are seen to be of two kinds. One is a definition, the other is the expression of a physical law. Also listed are the number of measures involved in the defining relation.

As previously mentioned (Gibbings 1980), this present approach differs from that of others. For example, Bridgman (1927) virtually proposed the definitions of all concepts to be inseparably linked with the definition of the measure of them as done here for only the 'derived' concept. It was because this approach raised difficulties for philosophers when considering the meanings of time and of extension that the present line of argument for primary quantities was previously advanced (Gibbings 1980).

A 'primary' concept is thus advanced as a sensed one, a 'derived' one is a definition of measurement in physics. This approach would appear to be consistent with Bridgman's view (1937, p 9) that '. . . operations which give meaning to our physical concepts

† See the table of notation at the end of this paper.

‡ A point also advanced by Esnault-Pelterie (1950, p 55).

should properly be physical operations, actually carried out...' as long as the operations leading to the present definitions of primary concepts are recognised as not, in the first instance, requiring numerical measure. It is relevant that Bridgman goes on to say (1937, p 10) that 'It must not be understood that we are maintaining that as a necessity of thought we must always demand that physical concepts be defined in terms of physical operations...'. This is consistent with the present distinction just mentioned, between defining relations formed from definitions and from physical laws.

The sixth logical step has now been discussed.

3. Combination of dimensions

It will be noticed from table 1 that all the defining relations, whether definitions or physical laws, are in the form of products of quantities: consequences of this now follow.

In physical analysis, to be meaningful, numerical addition of quantities is required to be of quantities measured in the same dimension and referred to a common unit quantity†. If this type of addition is to be as a multiplication then the multiplier must be a dimensionless number.

If different quantities of the same concept are multiplied then the constancy of relative magnitude will be retained; this is also true of a quantity raised to a power and of the product of the quantities of different concepts. If two physical quantities of different dimensions are multiplied then the idea of combined dimensions is introduced and this combination forms the dimensions of 'derived' concepts. It is then seen that multiplication and cancellation of dimensions are in accord with the dimensional description of 'derived' concepts. Cancellation of dimensions occurs, for example, when a velocity is multiplied by a time to represent distance, or $(\mathrm{L}/\mathrm{T})\mathrm{T} = \mathrm{L}$.

A further point requiring clarification is that this proportionality with unit quantity using linear scales only exists if those scales have a common origin for the zero value. This explains why temperature scales in Celsius and Fahrenheit cannot be compared for the purposes of dimensional analysis whereas those in Kelvin and Rankine can. The origin in this case does not have to be the so-called absolute zero: the Celsius and Reaumur scales can also be compared if we neglect the difference between the melting and triple points of water. Dimensional analysis requires use of linear scales and the combination of dimensions as products. Other definitions of 'derived' concepts that are unacceptable have been discussed previously and in detail (Taylor 1974, pp 3–4).

This section has amplified the listed logical steps (vii)–(ix).

4. Equality of dimensions

Usage of the equals sign has several meanings which include numerical equality, directional identity for a vector concept, and dimensional equality. The equals sign as a statement of the first of these does not necessarily imply the others. For example, in the relation for work, $Fs = \Delta E$, whilst there is a directional equality between F and s, such is not indicated by the equals sign. Or again, in an atmosphere at constant temperature, the pressure in atmospheres is given by

$$p = \exp(-gM_0z/RT)$$

† Esnault-Pelterie (1950) requires them to be of the 'same physical nature'; this seems much too restrictive.

and the dimensional equality is not satisfied. The restriction of addition to quantities of like dimension, as just proposed, results in dimensional equality of an equation; it is proposed here as a specification to be made when required and is then not a matter of proof but of statement.

The dimensions of 'derived' concepts are obtained in two ways. Referring again to table1, for item 1 the dimensions and the unit quantity of velocity are fixed by those of length and time. For item 3 the dimensions and unit quantity of the radiant energy are chosen arbitrarily, requiring the presence of Planck's constant. This constant then acts as a 'units conversion factor' so that dimensional equality is retained. This point has been discussed elsewhere. For example, as Klinkenberg (1968) has pointed out, in 'systems' of units the units conversion factors 'have been the means to make the system consistent'.

There is then the result that 'primary' unit quantities are used to quantify both primary concepts and also derived concepts obtained from defining relations without units conversion factors, whilst 'derived' unit quantities are used to quantify derived concepts obtained from defining relations with units conversion factors.

This discussion covers the listed logical steps (x) and (xi).

5. The complete equation

When an equation satisfies an equality of dimensions, the size of the unit quantity can be changed without correspondingly either introducing or removing a units conversion factor. If the unit quantity is changed by a factor then, for a linear scale, the equation is unchanged in algebraic form. Such is what has been called a 'complete equation' (Bridgman 1943).

Both Buckingham and later Bridgman used the expression 'complete equation'. The former used it to imply that no variables are omitted, presumably within the limits of precision with which the equation models the phenomenon (Buckingham 1914). He does, however, introduce the equality of dimensions as a separate idea. Bridgman in contrast clearly linked the notation of 'complete equation' with that of equality of dimensions and gives it the meaning, used here, that its algebraic form remains unchanged by changes in the size of the unit quantity. But Bridgman and other authors do not go as far as to specify a complete equation. For example, Bridgman and Sedov (1974) said that physical regularities (laws) are 'generally independent of the particular system of units of measurement selected from among a set of such systems .... That this should be so is plausible'. '... it is so exceedingly improbable as to be practically impossible ... if it did depend on one particular system of measurement'.

Bridgman's order of logic is different from that presented here in that he started with the acceptance of the complete equation, then derived the Pi-theorem—which is to follow here—and finally deduced equality of dimensions (Bridgman 1943, pp 37–41).

The manipulation of complete equations during the generation of solutions is limited, through the requirement to retain dimensional equality, to certain operational rules. This has been discussed in detail elsewhere (Bridgman 1959, Taylor 1974). Addition and multiplication are permissible but other functional operations, such as taking logarithms and forming binomial expansions, whilst generally acceptable numerically, dimensionally are only permissible if all arguments are non-dimensional:

this latter restriction is readily observed from the corresponding series expansions of these and like functions.

Bridgman's example in support of his discussion (1943) has been repeated by others but seems not to be a powerful one. He added $v = gt$ to $s = \frac{1}{2}gt^2$ to get $v + s = gt + \frac{1}{2}gt^2$, stating correctly that it is numerically correct. But the latter equation could be written as $v = v(s, g, t)$ which is analytically wrong, there being a surplus of independent variables; this point has been discussed elsewhere (Gibbings 1974).

It will be noticed that the present discussion does not specify the idea that it is essential that an equation must be a 'complete' one for it to represent validly a physical event; numerically this is not so. That idea was put forward by Buckingham and was supported by Bridgman. Here, the existence of a 'complete equation' is only stated as a consequence of the dimensional equality that is specified for the requirements of dimensional analysis.

These points amplify the logical steps (xii) and (xiii).

The problem of what variables are to be introduced into a functional relationship that represents a physical phenomenon, and before the Pi-theorem is used, is resolved by the following procedure.

(a) An understanding of a modelling of the physics enables listing of all the basic physical laws involved together with the boundary conditions to those equations; the laws must be expressed as 'complete' equations.

(b) Inspection of these equations and boundary conditions enables a listing to be made of all the variables and units conversion factors appearing in them.

(c) Where an equation listed at (a) is a defining relation for a derived quantity such as listed in table 1, the equation will contain the corresponding units conversion factor. There exists then, at choice, an ability to redefine the unit quantity of the derived concept in terms of those utilised in its derivation and hence deleting the units conversion factor from the list of (b) above.

Whilst this routine might appear to be straightforward there are problems in its application. These have been illustrated by examples elsewhere (Gibbings 1980, 1981).

The discussion now covers the listed logical steps (xiv) and (xv).

6. The Pi-theorem of regrouping

The principle theorem of dimensional analysis, known as the Pi-theorem, applies only to equations having equality of dimensions and subject to the operational rules described. It is now demonstrated anew and illustrated by specific examples.

For N variables, say $Q_1, Q_2, \ldots, Q_N$, there is

$$f(Q_1, Q_2, \ldots, Q_N) = 0 \tag{1}$$

which is to be a 'complete' equation. Each variable can have its dimensions expressed in terms of the dimensions, say $D_1, D_2, \ldots, D_n$, so that, according to the foregoing rules for combination of dimensions, a tabulation can be as in table 2. Here, the α_{ir} are non-dimensional numbers. For one or more i, some or all of α_{ir} could be zero.

The variable Q_i is now used to cancel the dimension in D_1; tabulation now is as in table 3. Equation (1) can be rewritten, within the prescribed operational limits to retain a 'complete' equation, as

$$f\{(Q_1^{\alpha_{i1}}/Q_i^{\alpha_{11}})Q_i^{\alpha_{11}}, \ldots, Q_i, \ldots, (Q_N^{\alpha_{i1}}/Q_i^{\alpha_{N1}})Q_i^{\alpha_{N1}}\} = 0$$

Table 2.

Variable	Dimension
Q_1	$D_1^{\alpha_{11}} D_2^{\alpha_{12}} \ldots D_n^{\alpha_{1n}}$
Q_2	$D_1^{\alpha_{21}} D_2^{\alpha_{22}} \ldots D_n^{\alpha_{2n}}$
$\vdots$	$\vdots$
Q_N	$D_1^{\alpha_{N1}} D_2^{\alpha_{N2}} \ldots D_n^{\alpha_{Nn}}$

Table 3.

Variable	Dimension
$Q_1^{\alpha_{i1}}/Q_i^{\alpha_{11}}$	$[D_2^{\alpha_{12}} \ldots D_n^{\alpha_{1n}}]^{\alpha_{i1}}/[D_2^{\alpha_{i2}} \ldots D_n^{\alpha_{in}}]^{\alpha_{11}}$
$\vdots$	
Q_i	$D_1^{\alpha_{i1}} D_2^{\alpha_{i2}} \ldots D_n^{\alpha_{in}}$
$\vdots$	
$Q_N^{\alpha_{i1}}/Q_i^{\alpha_{N1}}$	$[D_2^{\alpha_{N2}} \ldots D_n^{\alpha_{Nn}}]^{\alpha_{i1}}/[D_2^{\alpha_{i2}} \ldots D_n^{\alpha_{in}}]^{\alpha_{N1}}$

or

$$f_1\{Q_1^{\alpha_{i1}}/Q_i^{\alpha_{11}}, \ldots, Q_i, \ldots, Q_N^{\alpha_{i1}}/Q_i^{\alpha_{N1}}\} = 0. \tag{2}$$

Inspection of table 3 shows that Q_i is now the only variable in equation (2) that contains a dimension in D_1. The argument now is that, if the operational limits on the algebraic manipulation are satisfied, Q_i can only appear in such a way in equation (2) that if the dimensions cancel then so also can there be a numerical cancellation of Q_i. Thus equation (2) reduces to

$$f_2\{Q_1^{\alpha_{i1}}/Q_i^{\alpha_{11}}, \ldots, Q_N^{\alpha_{i1}}/Q_i^{\alpha_{N1}}\} = 0.$$

This process may be continued for successive cancellations of $D_2, D_3 \ldots D_n$ until there are no more dimensions left for cancellation, all the groups of variables then being non-dimensional: there is nothing mandatory about this completion, prior stages give equally valid equations.

If all the α_{ir} are non-zero then, with exceptions to be described, each time a cancellation is made:

(a) one variable is added, by multiplication, to each group of variables;

(b) one dimension is removed.

Thus from (b) there will be a total of n cancellations and after all these possible cancellations each group will contain $(1+n)$ variables and there will be $(N-n)$ groups. This is the present proof of the Pi-theorem.

If one of α_{ir} is zero, say α_{jr}, then the dimensions of the group with Q_j will not contain D_r. A cancellation will not be needed so that this group will contain only $(1+n)-1=n$ variables.

If, at the first set of cancellations†, the condition

$$\alpha_{i1}/\alpha_{ir} = \alpha_{j1}/\alpha_{jr} \tag{3}$$

† The order of the dimensions, D, is arbitrary.

for all $j = 1, 2, \ldots, N$ ($j = i$ is a trivial case) is satisfied, then this first cancellation will also cancel the dimension D_r. Thus now the number of groups will be $(N-n)-1 = N-n-1$ because there will be one fewer set of cancellations.

Before further discussing the foregoing proof its application is illustrated by examples.

For the impact force, F, of a jet of liquid at the position where the jet is breaking up we write, as a 'complete' equation,

$$f(F, \rho, V, l, \mu, \sigma) = 0. \tag{4}$$

The reduction of this equation to a function of non-dimensional groups follows the procedure of the preceding proof. It can be tabulated conveniently as in table 4.

Table 4.

K	F ML/T^2	ρ M/L^3	V L/T	l L	μ M/LT	σ M/T^2
1	F/ρ L^4/T^2		V L/T	l L	μ/ρ L^2/T	σ/ρ L^3/T^2
2	$F/\rho V^2$ L^2			l L	$\mu/\rho V$ L	$\sigma/\rho V^2$ L
3	$F/\rho V^2 l^2$ 1				$\mu/\rho V l$ 1	$\sigma/\rho V^2 l$ 1
k	3				3	3

Then from this table the reduction of equation (4) is finally to

$$f\{F/\rho V^2 l^2, \mu/\rho V l, \sigma/\rho V^2 l\} = 0. \tag{5}$$

Conclusions for this particular example are as follows.

(a) The number of variables, N, less the number of levels of cancellation, say K, equals the number of non-dimensional groups, say G. Or, $G = N - K$.

(b) The number of cancellations for each group, say k, is equal to the number of dimensions, n. Or, $k = n$ and also $k = K$.

(c) The number of variables in each group, say m, is given by $m = k + 1$.

(d) The value of K and hence of G is independent of the order of cancellation of the n dimensions because each dimension requires one cancellation in turn.

A second example is one that was given by Buckingham (1914) because it presents special difficulties. It is of the energy per unit volume, U, of an electromagnetic field. Proposing that

$$f(U, E, H, \varepsilon, \mu) = 0$$

gives the derivation in table 5.

The conclusions from this example are now that

(a) $G = N - K$

(b) $K < n$

(c) $k \leqslant K$

(d) $m = k + 1 \leqslant K + 1$.

Table 5.

K	U $\mathrm{M/LT^2}$	E $\mathrm{ML}/A\mathrm{T^3}$	H A/L	ε $A^2\mathrm{T^4/ML^3}$	μ $\mathrm{ML}/A^2\mathrm{T^2}$
1	U/μ A^2/L^2	E/μ A/T	H A/L	$\varepsilon\mu$ $\mathrm{T^2/L^2}$	
2	U/μ A^2/L^2	$E(\varepsilon/\mu)^{1/2}$ A/L	H A/L		
3	$U/\mu H^2$ 1	$(E/H)(\varepsilon/\mu)^{1/2}$ 1			
k	2	3			

Now the dimensions can be taken as effectively being those of the three cancelling quantities, that is

$$\mathrm{ML}/A^2\mathrm{T}^2, \qquad \mathrm{T^2/L^2}, \qquad A/\mathrm{L}$$

which can be simplified to

$$\mathrm{M/L^3}, \qquad \mathrm{T/L}, \qquad A/\mathrm{L}.$$

Again, the number of independent groups obtained is independent of the order of cancellation of the three dimensions.

From the foregoing general demonstration of the Pi-theorem, and the illustrative examples, general conclusions are:

(i) $G = N - K$
(ii) $m = k + 1$
(iii) $K \leqslant n$
(iv) $k \leqslant K$.

The discussion has now covered the logical step (xvi).

7. Existing proof of the Pi-theorem

The foregoing routine for reduction to non-dimensional groups, as distinct from the present proof of the Pi-theorem, is markedly similar to that described by Taylor (1974). Taylor made the important point that this method always results in a set of non-dimensional groups that is correct both in the number of groups and in the composition of each, and thus avoids the difficulty of some other methods in the prior determination of the number of independent dimensions (Taylor 1974, p 29) and of the permissible dimensions (Buckingham 1914, p 349 Ll 1–4). Thus existing methods have, for some examples such as the second one just given, raised practical difficulties in arriving at the correct number of non-dimensional groups, and so have led to means for determining these groups such as that described by Van Driest (1946). But, as Taylor has pointed out (1974, p 29), such discussions do not always lead to a determination that is straightforward.

As a proof, a comparison is now made of the present derivation with existing ones of which there are four basic variants.

A first version relies on the initial functional relation, such as that of equation (1), being in the form of a power product of the variables. This provision was used by Rayleigh (1885). This serious limitation was relaxed to a degree by Buckingham (1914), who specified that the function was to be represented by a finite series of power products. Both Focken (1953) and Massey (1971, p 55) justify this approach by reference to the Weierstrass approximation theorem, and so again this approach is approximate and requires the existence of a function that is continuous and so is not general. In presenting this version, Buckingham additionally separated all those variables having a common dimension and, dividing by one of them, expressed them as dimensionless ratios before proceeding to apply the Pi-theorem to the remaining variables. Neither the advantage nor the need for this procedure is clarified by such writers and is, indeed, not required by the present proof and method.

A second approach is that discussed, for example, by Birkhoff (1960, § 64) and is to express equation (1) as an infinite Maclaurin series. As not all examples are expressible in series form because of singularities, as indeed Birkhoff points out (1960, p 94), and as convergence of the series does not necessarily exist, there is again a limitation of application.

A third approach makes use of the concept of the invariance of a 'complete equation'. Birkhoff illustrates this (1960) by the example of the resistance of a closed body in a flow. Putting

$$D = f(\rho, V, l, \mu),$$

then

$$f\{D/\rho V^2 l^2, \mu/\rho V l, \rho, V, l\} = 0. \tag{6}$$

A change in the size of the unit quantity can then be made in turn so as to give a numerical value of unity to each of ρ, V and l. Taylor (1974) describes this as a use of units that are intrinsic to the problem rather than a use of extraneous units. It is then stated that equation (6) can be replaced by

$$f\{D/\rho V^2 l^2, \mu/\rho V l, 1, 1, 1\} = 0 \tag{7}$$

so that

$$f\{D/\rho V^2 l^2, \mu/\rho V l\} = 0.$$

This demonstration is basically that presented by Langhaar, who points out (1951, p 55) that his proof both limits the independent variables to having positive values, and (Langhaar 1951, p 58) requires that in the final formulation a Pi-group is to be a single-valued function of the other Pi-groups: neither of these serious limitations appears in the present proof.

A further difficulty with this third approach arises from the transformation between equations (6) and (7). An equation that represents a real physical event is understood to be a description of the relation between variables that retains truth as these variables change in numerical value, these values being related to fixed values of each and every unit quantity: it is a matter of semantics. A fixed linear transformation applied to such an equation retains this meaningful representation by the transformed equation. But the transformation between equations (6) and (7) is one that must change continuously as the last three variables of equation (6) are each continuously changed: this has the effect of continuously changing the unit quantity of these variables. Then the representation is no longer meaningful as just described and also a constancy of relative magnitude is not retained.

Finally, a fourth approach, such as by Bridgman (1959), relies on the function of equation (1) being differentiated in turn with respect to each and every one of the variables. But the differentiability of a function is not generally demonstrable. Esnault-Pelterie (1950) advanced further detailed criticisms of Bridgman's proof by use of an example ascribed to Villat.

8. Concluding comments

The logical steps in dimensional analysis, from the basic idea of definition of a primary concept up to the Pi-theorem, are set out. In general, the existing literature does not set forth a complete listing of logical steps and, further, where some steps have been delineated they differ in the order from that proposed here. Certain steps here differ from other presentations; for example, the idea of distinguishing between primary and derived concepts, the former being helpful to philosophy; also the idea that dimensional equality is something not to be inferred but to be specified for the purposes of dimensional analysis.

Finally, a derivation of the Pi-theorem is given that avoids limitations of existing methods. The routine of operation of the theorem is very similar to an existing one and is straightforward even in otherwise difficult examples. Experience has shown that both this proof of and the routine for the Pi-theorem are acceptable for use at an elementary stage of instruction whereas many existing texts, even specialised ones on dimensional analysis (Massey 1971, Pankhurst 1964), often omit a proof, quoting only the result of this theorem.

Acknowledgment

The critical assessment of an early draft of this paper by Dr C J S Clarke gave much of this paper such clarity as it has.

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Notation

SI quantities:

M	mass	A	current
L	length	C	light intensity
T	time	n	amount
θ	temperature	α	angle

Symbol	Meaning	Symbol	Meaning
a	Acceleration	P_0	Mechanical equivalent of light
$\bar{c}$	Mean velocity	q	Charge
D	Drag	Q	Heat, quantity
D_i	Dimensions	Q_i	Variables
E	Electric field	r	Radius
E_k	Kinetic energy	R	Gas constant
E_l	Luminous energy	s	Distance, length
E_r	Radiant energy	t	Time
f	Frequency	T	Temperature
F	Force	v	Velocity
g	Gravitational acceleration	U	Energy of electromagnetic field
g_0	Inertia constant	V	Velocity
G	Number of non-dimensional groups	z	Height
h_p	Planck constant	α_{ir}	Index
H	Magnetic field strength, power	α	Angle
i	Current	β_0	Angle constant
J	Mechanical equivalent of heat	γ	Gravitational constant
k	Number of cancellations for a group	ε	Permittivity
k_B	Boltzmann constant	ε_0	Permittivity of space
K	Number of levels of cancellation	Γ	Amount of gravity
l	Length	λ	Luminous flux
m	Mass	μ	Coefficient of viscosity, magnetic permeability
m	Number of variables in a group		
M_0	Molecular mass	μ_0	Magnetic permeability of space
n	Number of dimensions	ρ	Density
N	Number of variables	σ	Surface tension
p	Pressure		